

Proximate Time-Optimal Reference Tracking of a Harmonic Oscillator

Roger A. Braker¹ and Lucy Y. Pao²

Abstract—Motivated by a desire to use compressed sensing in Atomic Force Microscopy, we revisit the Proximate Time-Optimal Servomechanism (PTOS ω) regulator we developed previously for the harmonic oscillator, giving a more detailed description of its construction and show new experimental results. Then, we extend that previous work to include set-point tracking. We show that, in contrast to regulator control, the time-optimal switching curve becomes asymmetric and in fact the problem can be recast as a regulator with asymmetric actuator limits. We then develop a PTOS ω approximation and present simulation and experimental results.

I. INTRODUCTION

The traditional Atomic-Force-Microscope (AFM) imaging method is to raster scan the sample with an atomically sharp probe. Raster scanning is not only a very time consuming process, it can also lead to damaging either the sample or the AFM probe tip or both. Traditionally, efforts to increase imaging speed have focused on increasing the raster scan rate via advanced control algorithms [1], [2].

More recently, researchers have begun to investigate using more efficient scan patterns since often the raster scan spends significant time over uninteresting regions [3].

In a similar vein, [4] suggested a method of AFM imaging whereby a random sample of point-to-point measurements are taken. The sample topology is then reconstructed using the theory of compressed sensing [5]. Taking this random sample of point-to-point measurements reduces the tip-sample interaction and can improve the integrity of both the sample and AFM tip.

Minimizing the imaging time in this scheme requires that the rest-to rest maneuver times between point measurements be minimized. AFM stages are typically actuated with piezoelectric actuators. The linear dynamics of these actuators are characterized by a series of resonances and anti-resonances which lack of a rigid body mode [6]. This can also be seen in the frequency response of the AFM stage considered in [1]. Hence, even the simplest model for such a stage will be second-order with complex poles. This contrasts with many other mechatronic systems like hard-disk drives which are often reduced to the simplified model of $G(s) = a/s^2$.

For a linear time-invariant system, Pontryagin's Maximal Principle leads to a time-optimal control that is bang-bang [7]. Excellent resources for synthesizing these controls as

feedback laws can be found in [8]–[10]. Unfortunately, the bang-bang feedback control is impractical. In any real control system, there will be process and measurement noise, uncertainty in the system parameters and imperfect actuators which will cause the control to chatter between its maximum and minimum value.

A large body of work exists which develops methods to combat this chattering problem. However, much of it focuses on rigid body systems. In [11], a robust control law for a double integrator plant is developed using sliding mode techniques. Other researchers have developed the Proximate Time-Optimal Servomechanism (PTOS) for a double integrator plant [12] and a triple integrator plant [13].

Reference [14] suggests pre-computing the time-optimal trajectory which is then tracked with a stabilizing trajectory tracking control law. At the end of this reference trajectory, an end-game control law is implemented to eliminate any final error. A downside to this method is it requires pre-computation of individual trajectories for each initial condition and target state.

In [15], we developed a near time-optimal regulator controller for a system with purely complex eigenvalues. In contrast to the controllers for rigid body plants, this control law is not immediately extendable to set-point tracking. To implement set point tracking for those systems, the switching curve, and hence the PTOS “tube” is simply translated in the phase plane. As we discuss in Section IV, for the simple oscillator system performing reference tracking, not only does the switching curve become asymmetric but our control law must be modified to provide a constant input at steady state.

This paper is organized as follows. In Section II we give review our previous work with additional details of the method for developing the PTOS ω regulator in order to help guide our development of the reference tracking PTOS ω . We follow this discussion with some experimental results of the regulator in Section III. We show in Section IV the time-optimal solution for a holdable set-point is governed by an asymmetric switching curve. We show this curve is a translated copy of the regulator switching curve for a system with asymmetric control limits, which in fact, is the switching curve in the error coordinates. This observation helps us to construct the reference tracking controller as a feedback control law. Using this development, we then extend the PTOS ω controller of Section II to the case of set-point tracking in Section V and give a discussion of its stability in Section VI. In Section VI-A, we show experimental results for this set-point tracking PTOS ω and provide concluding remarks with a short discussion of the

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R.A. Braker is a graduate student and L.Y. Pao is the Richard & Joy Dorf Professor; both are with the Dept. of Electrical, Computer, and Energy Engineering at the University of Colorado, Boulder, CO 80309, roger.braker@colorado.edu, pao@colorado.edu.

case with damping in Section VII

II. OVERVIEW OF REGULATOR PTOS ω

Suppose we have a second order system with complex eigenvalues, described in state space by

$$\dot{x} = Ax + Bu(t) \quad (1)$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad (2)$$

$$B = \begin{bmatrix} 0 \\ b_o \end{bmatrix} \quad (3)$$

It will be useful to define $c = \frac{b_o}{\omega^2}$. The system is driven by a bounded control¹,

$$u \in [-1 \ +1]. \quad (4)$$

In [15] we modified the time-optimal regulator controller into a near near time-optimal regulator, which we called the PTOS ω for which we proved stability given an initial state anywhere in \mathbb{R}^2 . For brevity, here we assume that the initial state is a holdable equilibrium state since ultimately we are interested in rest-to-rest movements. These holdable equilibria are given by $x_{eq} = -A^{-1}Bu$, or

$$x_{2,eq} = 0 \quad (5)$$

$$x_{1,eq} \in [-c \ c]. \quad (6)$$

$$(7)$$

In general the time-optimal feedback control law is characterized by a switching curve, which in this case, is comprised of an infinite sequence of half ellipsis. The first set of such ellipses give the control law as [9]

$$u = \text{sgn}(f_{to}(x_1) - x_2) \quad (8)$$

$$f_{to}(x_1) = -\text{sgn}(x_1)\omega\sqrt{2cx_1 - x_1^2}, \quad |x_1| \leq 2c \quad (9)$$

$$\text{sgn}(\xi) = \begin{cases} -1 & \xi < 0 \\ 0 & \xi = 0 \\ +1 & \xi > 0. \end{cases} \quad (10)$$

A. Development of $f_p(x_1)$

Consider the controller given by,

$$u_p(t) = \text{sat}[k_2(-x_2 + f_p(x_1))]. \quad (11)$$

It will be useful for the following discussions to make use of the following divisions of the state space:

$$\mathcal{T} = \{x: |-x_2 + f_p(x_1)| \leq \frac{1}{k_2}, \quad x_1 \in [-2c, 2c]\} \quad (12)$$

$$\mathcal{B} = \{x: x \in \mathcal{T}, \quad x_1 \in [-c, c]\} \quad (13)$$

$$\mathcal{L} = \{x: x \in \mathcal{B}, \quad x_1 \in [-x_\ell, x_\ell]\} \quad (14)$$

$$\mathcal{U}^- = \{x: x \notin \mathcal{T}, \quad x_2 > -f_p(x_1)\} \quad (15)$$

$$\mathcal{U}^+ = \{x: x \notin \mathcal{T}, \quad x_2 \leq -f_p(x_1)\}. \quad (16)$$

¹It is quite unlikely that a particular system will saturate at ± 1 . This convention is made only for convenience here and such systems can be easily dealt with changing by b_o to $\bar{b}_o = b_o M$ where M is the new saturation limit.

where $0 < |x_\ell| < c$ and \mathcal{U}^+ and \mathcal{U}^- are the regions of the state space which result in a saturated control, while $x \in \mathcal{T}$ results in an unsaturated control and $\mathcal{L} \subset \mathcal{B} \subset \mathcal{T}$.

By making $f_p(x_1)$ approximate $f_{to}(x_1)$ and making k_2 large, then in (11) we have a finite slope approximation to (8). Furthermore, we can employ a linear feedback controller near the origin by defining f_p as the continuously differentiable (for $x_1 \in (-2c, 2c)$) function, defined piecewise as

$$f_p(x_1) = \begin{cases} f_\ell(x_1) & |x_1| \leq x_\ell \\ f_{n\ell}(x_1) & x_\ell < |x_1| < 2c \end{cases} \quad (17)$$

where we require that

$$f_\ell(x_\ell) = f_{n\ell}(x_\ell) \quad (18)$$

$$f'_\ell(x_\ell) = f'_{n\ell}(x_\ell). \quad (19)$$

By making $f_\ell(x_1)$ a linear function of x_1 , then for $x_1 \in \mathcal{L}$, (11) describes the familiar equation for linear state feedback with the $\text{sat}(\cdot)$ function enforcing respect for the control limits. Specifically, define the linear portion of f_p as

$$f_\ell(x_1) := -\left(\frac{k_1}{k_2}\right)x_1, \quad |x_1| < x_\ell. \quad (20)$$

We construct the entire f_p by connecting this linear f_ℓ to vertical translations of f_{to} so such that (18) and (19) are satisfied. Taking the Taylor approximation of (9) about x_ℓ yields

$$f_{to}(x_1) \approx -\frac{1}{k_2} - \frac{k_1}{k_2}x_1 \quad (21)$$

where

$$k_2 = \frac{\sqrt{2cx_\ell - x_\ell^2}}{\omega cx_\ell} \quad \text{and} \quad k_1 = \frac{c - x_\ell}{cx_\ell}. \quad (22)$$

Since the switching curve must go through the origin, add the x_2 -intercept, $\frac{1}{k_2}$, to (21) to yield $f_\ell(x_1)$. We connect the parts of f_{to} outside \mathcal{L} to f_ℓ by shifting the right half of f_{to} up by $\frac{1}{k_2}$ and the left half down by $\frac{1}{k_2}$. This idea is shown in Figure 1 which shows a plot of $f_\ell(x_1)$ and $f(x_1)$. We thus obtain

$$f_\ell(x_1) = -\frac{k_1}{k_2}x_1 \quad (23)$$

$$f_{n\ell}(x_1) = \text{sgn}(x_1) \left[-\omega\sqrt{2c|x_1| - x_1^2} + \frac{1}{k_2} \right]. \quad (24)$$

Note that if $x_\ell \rightarrow 0$ then $k_2 \rightarrow \infty$ and $\frac{1}{k_2} \rightarrow 0$ and we recover (8).

Theorem 1: The control law described by (11), (23) and (24) leads to a stable closed loop system.

Proof: For the full proof, see our previous work in [15]. However, the features relevant to this paper are that, provided we restrict $|x_\ell| < 0.4c$: (1) states within region \mathcal{B} remain there which we showed by examining the time derivative of $u(t)$ along the upper and lower boundaries of \mathcal{B} ; and (2) states in \mathcal{B} tend asymptotically to the origin which we showed by giving a Lyapunov function for region \mathcal{B} . ■

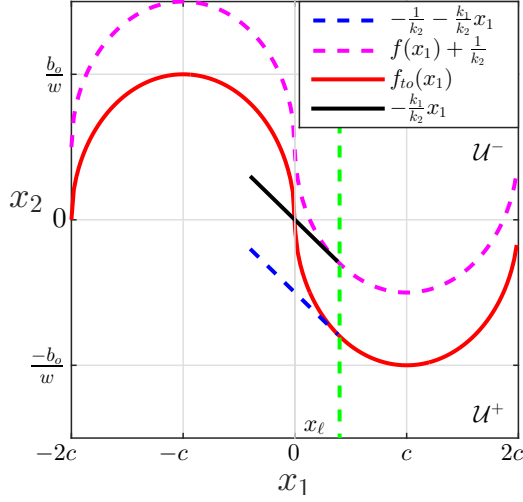


Fig. 1: Illustration of the development of $f_p(x_1)$. The red curve depicts $f(x_1)$ and the blue-dashed line is the Taylor approximation of $f(x_1)$ about x_ℓ . Both of these are shifted down by $\frac{1}{k_2}$ to give f_ℓ (solid black), and f_{nl} (dashed-pink). Note that generally when we plot the switching curve, we plot $-f(x_1)$ while here we have plotted $+f(x_1)$.

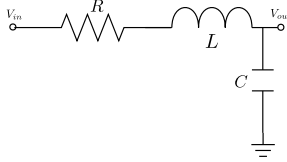


Fig. 2: Schematic of the LRC circuit used to test the control law.

III. EXPERIMENTAL RESULTS

We tested the control law (11) on the LRC circuit shown in Figure 2 which is described by the transfer function

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \quad (25)$$

If we could construct this circuit with $R = 0$, we would have exactly the plant the described in (2). Of course, this is impossible but we can get close. We constructed this circuit with a capacitor and inductor which have the nominal values $C = 0.235\mu\text{F}$ and $L = 100\text{ mH}$. The inductor has an internal resistance of $R = 82\Omega$.

We performed a white noise system identification which yielded

$$G(s) = \frac{4.478e7}{s^2 + 743.5s + 4.449e7}. \quad (26)$$

Thus, $\zeta = 0.063$, $\omega = 6669.7\text{rad/s}$, $b_o = 4.478e07$. We programmed the control law in (11) into a National Instruments Compact RIO (NI cRIO-9082) FPGA, using a sample rate of 100KHZ. Because our control law utilizes both states of the system, we implemented a digital prediction observer on the FPGA to provide an estimate of x_2 .

Because the control law utilizes non-linear feedback, analyzing the convergence of the observer and plant becomes

problematic. Namely, the separation principle no longer holds. Proceeding anyways, we chose observer gain $L = [0.440 \ 4663.6]^T$ which, in the linear case, would place the observer poles at four times the natural frequency of the plant and increase the damping ratio to $\zeta = 0.95$.

We induced an initial condition of $V_{in} = 1\text{ [V]}$ to the system. After a settling period, we turned on the PTOS ω controller. For this experiment we set $x_\ell = 0.1c$. The results of this experiment are displayed in Figures 3 and 4 where it can be seen that even for a plant with small but non-zero damping, our control law performs well, with both the simulation and experiment agreeing nicely.

IV. REFERENCE TRACKING TIME-OPTIMAL FEEDBACK CONTROLLER

Suppose that rather than time-optimal *regulation*, we would like our plant to track *setpoint* changes as fast as possible. For plants with complex eigenvalues, this new requirement changes the structure of the switching curve giving it an asymmetry. This is in sharp contrast to rigid body plants where the switching curve is simply shifted in the phase plane. This difference makes set point tracking in both the time-optimal and PTOS ω case more difficult than it is for rigid body plants since the switching curve becomes asymmetric and changes for every new setpoint.

To show how this difference comes about, we review the the standard derivation of the switching curve [9]. We minimize the cost

$$J = \int_0^t 1 d\tau \quad (27)$$

which gives the Hamiltonian

$$H = 1 + \lambda^T (Ax + Bu) \quad (28)$$

where $\dot{\lambda} = A^T \lambda$ defines the adjoint dynamics. Given the boundedness of $u(t)$, the minima of H are given by

$$u = -\text{sgn}(\lambda^T B) = \lambda_2(t)b_o. \quad (29)$$

Thus, the control switches at times t_s when the costate $\lambda_2(t)$ vanishes. For the first and last switch, this time is unknown since theses roots depends on the boundary conditions for $\lambda(t)$ which are unknown.² However we make the observation that, since the costate rotates in the phase plane at a rate of ω , no switch can last longer than π/ω [s]. Thus, we can locate the final leg of any time optimal trajectory by integrating the system in (2) backwards for π/ω units of time with $u = \pm 1$. This is the curve labeled f_{to} in Figure 1. Since the control is bang-bang, trajectories approaching this curve must be under the control $u = \mp 1$, and thus the control switches upon the state reaching the curve, prompting us to call it the *switching curve*. Repeating the process of backwards integration starting from the first segment of f_{to} locates the next segment of the switching curve. However, since the first segments are half ellipses with length $2c$, we

²the difficulty of solving the two point boundary value problem defined by $\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ A^T \lambda \end{bmatrix}$ arises because (greedily!) $x(t)$ has take all four boundary conditions and there are none left for $\lambda(t)$.

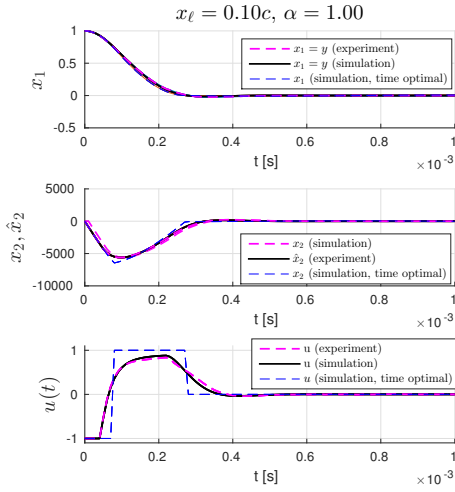


Fig. 3: Comparison between simulation and experimental results. The controller from both simulation and experiments is driven by the states from a digital observer. Note that in the time series plot of x_2 , we have plotted the estimate, \hat{x}_2 for the experimental case since this is the only data available.

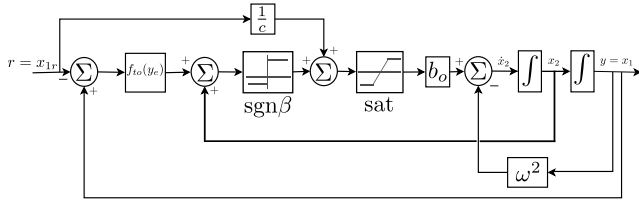


Fig. 5: Block diagram of time-optimal controller.

need only concentrate on these first segments since we are interested in holdable equilibria, $x_{1eq} \in [-c, c]$.

The typical derivation of the switching curve performs this process beginning at the origin [8]–[10], [16], which yields the regulator of (8). To derive the curve for target states other than the origin, we must integrate backwards starting at x_r . However, to formulate this as a feedback control law, it is more instructive to examine the system in the error coordinates, $x_e = x - x_r$. First, note that when the $x_e = 0$, we must apply the constant control

$$u_{ss}(t) = \frac{x_{1r}}{c}, \quad \forall t \geq t_f \quad (30)$$

if we are to hold the state at x_r . The error dynamics are described by

$$\dot{x}_e = Ax_e + Ax_r + Bu \quad (31)$$

$$= Ax_e + \begin{bmatrix} 0 \\ b_o \end{bmatrix} \left(u - \frac{1}{c}x_{1r}\right). \quad (32)$$

Thus, driving the system to $x(t > t_f) = x_r$ is equivalent to driving the error state x_e to the origin *if the error dynamics*

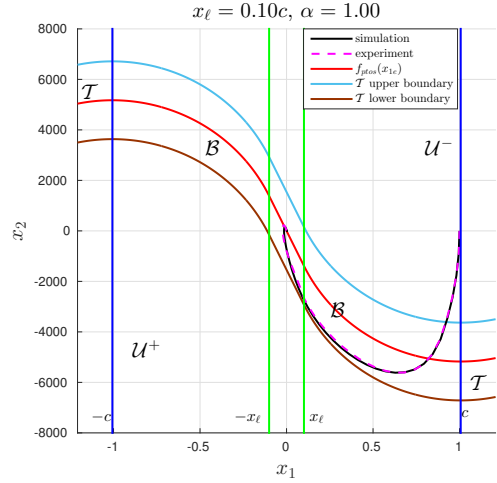


Fig. 4: Phase-plane trajectories of simulated and experimental system.

are driven by an assymetric control $\bar{u}(t)$, namely

$$\bar{u}_{min} = -1 - \frac{1}{c}x_{1r} \quad (33)$$

$$\bar{u}_{max} = 1 - \frac{1}{c}x_{1r} \quad (34)$$

We can construct the time-optimal switching curve for the error coordinates by solving (32) backwards in time from the origin of the error phase plane which will give

$$x(0) = \bar{u}_{min} \int_T^0 \Phi(0, \tau) B d\tau \quad (35)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \bar{u}_{min/max} c (1 - \cos \omega(-T)) \\ \frac{b_o \bar{u}_{min/max}}{\omega} \sin \omega(-T) \end{bmatrix} \quad (36)$$

for $T \leq \pi/\omega$. By eliminating time, we can solve for x_2 as a function of x_{1e} which is $f_{to,e}$,

$$f_{to,e}^- = -\text{sgn}(x_{1e})\omega \sqrt{2x_{1e}c\bar{u}_{min} - x_{1e}^2}, \quad x_{1e} < 0 \quad (37)$$

$$f_{to,e}^+ = -\text{sgn}(x_{1e})\omega \sqrt{2x_{1e}c\bar{u}_{max} - x_{1e}^2}, \quad x_{1e} \geq 0. \quad (38)$$

$$(39)$$

We use this convention that the superscript “−” denotes quantities of the LHP switching curve and “+” for the RHP switching curve through the rest of this paper. This switching curve is plotted in Figure 6. It is worth pointing out that the same result, less a translation of x_{1r} , will be obtained by generating the switching curve in the unshifted coordinates by integrating (2) backwards from x_r .

Note that the center of each ellipse is located at $c\bar{u}_{min}^{max}$ and the semimajor axis of each ellipse is also $c\bar{u}_{min}^{max}$ so that each ellipse has a different size and a different center. We

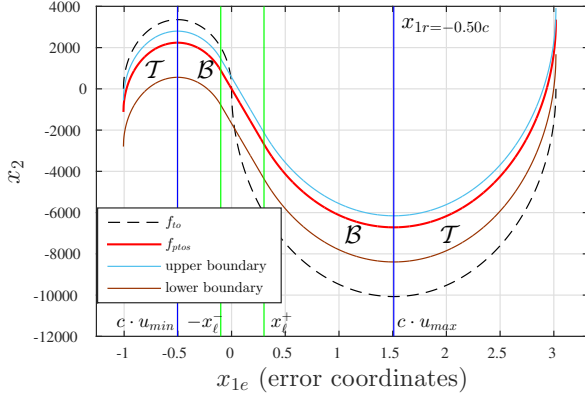


Fig. 8: Setpoint tracking $f_{p,ref}$ curves and saturation region

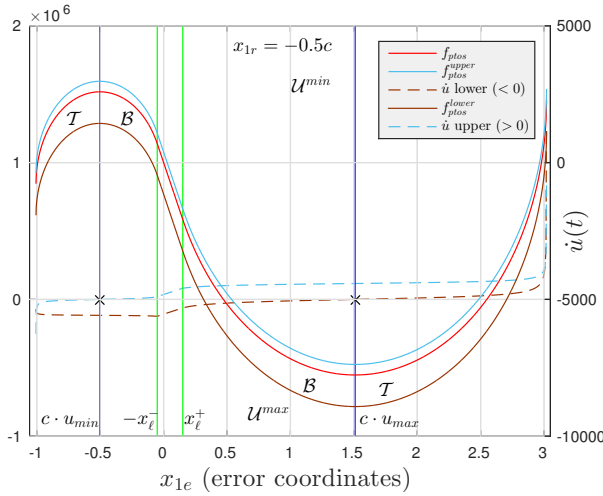


Fig. 9: This figure shows \dot{u} evaluated along the upper and lower saturation boundaries, for $x_{1r} = -0.4c$ and $\lambda = 0.1$. Since for the state to remain trapped in \mathcal{B} , we need $\dot{u}_{\ell} < 0$ and $\dot{u} > 0$, this plot shows \mathcal{B} is invariant.

does not exit via the saturation boundaries, we consider the time derivative of $u(t)$ along the boundaries of \mathcal{B} which is

$$\dot{u} = f'_p(x_1)\bar{u}_{min} + k_2 f'_p(x_1)f_p(x_1) + k_2 \omega^2 x_1 - k_2 b_o \bar{u}_{min}^{max} \quad (52)$$

where $\dot{u} > 0$ along the upper boundary and $\dot{u} < 0$ along the lower boundary implies that the state does not exit.

Using these parameters and a forward difference approximation to calculate f'_{pe} , we can plot (52) evaluated along both the upper and lower boundaries. This is shown in Figure 9. Indeed, we see that \dot{u} evaluated at the upper boundary becomes negative at $c\bar{u}_{min}$. Similarly, \dot{u} evaluated along the lower boundary becomes positive at $c\bar{u}_{max}$ which indicates that $x_e \in \mathcal{B}$ remain in \mathcal{B} . This is analogous to the regulator case where we showed the state exits the lower boundary in the right half plane for $x_1 > c$ and the upper boundary in the left half plane for $x_1 < -c$.

To see that $x_e \in \mathcal{B}$ tends asymptotically to the origin (of

the error space), we use the Lyapunov function

$$V(x_e) := \frac{1}{2}x_e^2 + \int_0^{x_{1e}} p(s)ds \quad (53)$$

$$p(x_1) := \omega^2 x_{1e} - b_o k_2 f_p(x_{1e}) \quad (54)$$

The only difference between this $V(x)$ and the one we used in [15] is a change in sign that accounts for the definition of f_p differing by a negative sign. Thus we refer the reader to that reference for the proof that $V(x)$ is indeed a Lyapunov function.

A. Reference tracking PTOS ω results

Using the same experimental setup as discussed in Section III, we tested the reference tracking PTOS ω controller. The time histories of these results are shown in Figure 10 along with the simulated time-optimal trajectories for comparison. Figure 11 shows the phase-plane plots for the simulated and experimental systems. As predicted in Section VI, the states do indeed remain trapped in \mathcal{B} , notably not exiting the lower boundary.

VII. CONCLUSIONS AND REMARKS ON DAMPING

In this paper we have given a detailed derivation of the PTOS ω that we then used to extended the PTOS ω to include reference tracking. We also verified these controllers in both simulation and physical experiment.

As we did not provide a full stability proof for the reference tracking PTOS ω , this is certainly an area for future work.

To properly include damping will be significantly more challenging. The switching curve in that case is described by a logarithmic spiral. Though it is still possible to eliminate time from the equations as we did with (36), it is not possible to isolate x_1 or x_2 on one side. However, the switching curve is easily generated via backwards integration and can then be represented in a lookup-table. The techniques discussed in Sections II-A and V can be used to create a “numeric” PTOS controller, e.g. by numerically calculating the derivatives of f_{to} to generate k_1^{\pm} and k_2^{\pm} . The challenges persist however since f_{to} is represented in a lookup-table and f_{to} changes for every reference input.

For our use case, of visiting a pre-determined set of points for a compressed sensing scheme, it is also possible to generate a family of switching curves offline and then program the controller to download the next needed curve while the current set point is being tracked.

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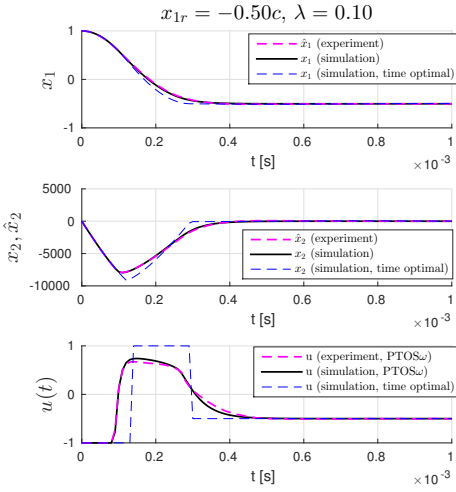


Fig. 10: Comparison between simulation and experimental results for the reference tracking $\text{PTOS}\omega$. The initial condition is set to $x_1 = 1$ and the reference is $x_{1r} = -0.5c$. The controller from both simulation and experiments is driven by the states from a digital observer. Note that in the time series plot of x_2 , we have plotted the estimate, \hat{x}_2 for the experimental case since this is the only data available.

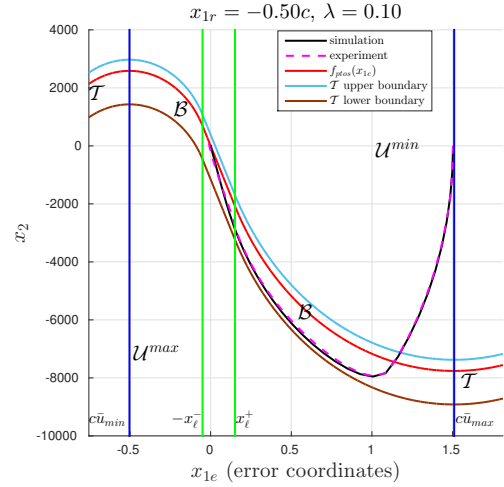


Fig. 11: Phase-plane trajectories in the error coordinates of both the simulated and experimental systems.

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